The η-Invariant as a Lagrangian of a Topological Quantum Field Theory

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Contents

1 The category

A topological quantum field theory (TQFT) starts from the categories C of oriented compact n-dimensional manifolds (M, N) with boundary $N = \partial M$ and ∂C of closed $(n-1)$-dimensional manifolds. The morphisms are orientation and boundary preserving diffeomorphisms. Let W denote the category of finite-dimensional complex Hilbert spaces. The morphisms in W are the isometries. Defining a TQFT is specifying a functor $H : \partial C \to W$ as well as a section E of the composition $H \circ \partial$, where $\partial : C \to \partial C$ is the functor of taking oriented boundaries. For the convenience of the reader we recall the notion of a section of a functor.

Definition 1.1 *Let $F : A \to B$ be a functor between the categories A, B. A section $E \in \Gamma(F)$ of F associates to any object $X \in A$ an element $E(X) \in F(X)$ such that for any morphism $a : X \to Y$ in A we have $F(a)E(X) = E(Y)$.*

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The notion of a section of a functor is similar to the definition of a parallel section of a flat vector bundle. H associates functorially to any oriented closed $(n-1)$-dimensional manifold $N \in \partial C$ a finite-dimensional complex Hilbert spaces $H(N)$. One requires that H satisfies the following axioms.

Axiom 1.2 (orientation) $H(-N) = H(N)^v$

Here V^v denotes the dual space of V and $-N$ is N with the opposite orientation. One requires that the equality in the axiom is functorial, i.e. for an orientation preserving diffeomorphism $f : N_0 \to N_1$ we have

$$\begin{array}{ccc} H(-N_0) & = & H(N_0)^v \\ H(F)\downarrow & & H(F)^v\uparrow \\ H(-N_1) & = & H(N_1)^v, \end{array}$$

where the horizontal equalities are given by the axiom.

Axiom 1.3 (additivity) $H(N_0 \oplus N_1) = H(N_0) \otimes H(N_1)$.

Here $N_0 \oplus N_1$ denotes the disjoint union of manifolds. Again this identifiction is required to be compatible with Axion 1.2 and to be functorial, i.e. for orientation preserving diffeomorphisms $f : N_0 \to \bar{N}_0$ and $g : N_1 \to \bar{N}_1$

$$\begin{array}{ccc} H(N_0 \oplus N_1) & = & H(N_0) \otimes H(N_1) \\ H(f,g)\downarrow & & H(f) \otimes H(g)\downarrow \\ H(\bar{N}_0 \oplus \bar{N}_1) & = & H(\bar{N}_0) \otimes H(\bar{N}_1). \end{array}$$

By convention $H(\emptyset)$ is canonically identified with $\mathbf{C}$.

The section E associates to every compact oriented n-dimensional manifold $(M, N) \in C$ with boundary N a vector $E(M, N) \in H(N)$ satisfying the orientation, additivity and the locality axioms.

Axiom 1.4 (orientation)

$$Tr E(-M, -N) \otimes E(M, N) = 1,$$

where $Tr : H(-N) \otimes H(N) \to \mathbf{C}$ is the pairing given by Axiom 1.2.

Axiom 1.5 (additivity)

$$E((M_0, N_0) \oplus (M_1, N_1)) = E(M_0, N_0) \otimes E(M_1, N_1)$$

Here the vectors are compared using the identification $H(N_0 \oplus H_1) = H(N_0) \otimes H(N_1)$. In order to state the locality axiom note that if two compact oriented manifolds M_0, M_1 with boundary have a boundary component N, $-N$, respectively (here we fix the identifications $N \to M_i$, $i = 0, 1$), we can glue M_0 and M_1 along N obtaining a new compact oriented manifold manifold $M := M_0 \sharp_N M_1$. The glueing does not

affect the other boundary components. Thus, the boundary of M consists of the union of the boundary components of M_0 and M_1 different from $-N, N$. Using the orientation axiom and the additivity axiom for H we get a natural map

$$Tr : H(\partial M_0) \otimes H(\partial M_1) \to H(\partial M)$$

, which contracts the pair $H(N) \otimes H(-N)$ inside $H(\partial M_0) \otimes H(\partial M_1)$. We will not always use the pair notation (M, N), in particular if more than one boundary component is involved.

Axiom 1.6 (locality)

$$E(M_0 \sharp_N M_1) = Tr\, E(M_0) \otimes E(M_1)$$

The use of that functorial language is very convenient since we have to take into account very carefully the automorphisms of the objects involved. Indeed, there will be a great difference between an isomorphism and a canonical isomorphism. All the theory is based on this fact.

The axioms stated above are the Atiyah-Segal-Axioms of a TQFT [1], see also the papers of Freed [5] and Freed/Quinn [6]. Of special interest are the values of E on closed manifolds, which are complex numbers. By their very definition they are diffeomorphism invariants.

We are going to use the η-invariant of twisted signature operators in order to define a local Lagrangian of a TQFT. Since the η-invariant depends heavily on the choice of Riemannian metrics we will use a difference construction. By the local variation formula the metric dependence will drop out. This involves flat bundles, which therefore have to be included into the category. Thus, we consider the category D consisting of compact oriented n-dimensional manifolds (M, N, F) with boundary N equipped with a flat Hermitian vector bundle F. Analogoulsy, ∂D consists of pairs (N, F_N), where N is an $(n-1)$-dimensional closed oriented manifold and F_N is a flat Hermitian vector bundle over N. The morphisms in both categories are pairs (f, Φ) of orientation preserving diffeomorphisms f and isomorphisms Φ of flat Hermitian bundles over f, i.e.

$$\begin{array}{ccc} F_0 & \xrightarrow{\Phi} & F_1 \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{f} & M_1 \end{array} .$$

A local Lagrangian of a TQFT looks similar to a TQFT itself. It is given by a functor $H : \partial D \to V$ and a section $E \in \Gamma(H \circ \partial)$ satisfying the orientation, additivity and locality axioms stated above (modified in the obvious way for D, ∂D). Note that the local Lagrangian of a TQFT, we will define, satisfies the orientation axiom up to a factor ± 1. This can be avoided by squaring everything.

2 The η-invariant of twisted signature operators

In this section we describe the η-invariant of twisted signature operators on oriented Riemannian manifolds of dimension $n = 4k - 1$ with boundary. It depends on the choices of a boundary condition and a Riemannian metric. One result of [3] was to clarify the dependence of the η-invariant on these choices, in particular on the boundary condition. These results will later be used to make things independent of the boundary condition and the choice of metrics.

Let (M, N) be an $n = 4k - 1$-dimensional Riemannian manifold with boundary. We always assume that the Riemannian metric g of M has a product structure near N, i.e. g is of the form $g = (dr)^2 + g^N$, where r is the outer normal coordinate to N and g^N is a Riemanian metric on N independent of r.

Let $F \to M$ be a flat Hermitian vector bundle over M. Then we can consider the twisted signature operator D_F acting on sections of $\Lambda^{ev}T^vM \otimes F$. Let $\omega \otimes f$ be a local section of that bundle with $\nabla^F f = 0$ and ω a p-form, then

$$D_F(\omega \otimes f) = (-1)^{p+k}((*d - d*)\omega) \otimes f.$$

If F is trivial one-dimensional we denote the corresponding operator, called the odd signature operator in [3], by D. In a neighbourhood of the boundary N of M the operator D_F has the structure

$$D_F = I(\frac{\partial}{\partial r} + D_{F_N}),$$

where we consider the sections of $\Lambda^{ev}T^vM \otimes F$ as r-dependent sections of the restriction of that bundle to N and D_{F_N} is an elliptic differential operator acting on sections of that restriction. I is a bundle endomorphism of $(\Lambda^{ev}T^vM \otimes F)_{|N}$ satisfying $I^* = -I$, $I^2 = -1$ and $ID_{F_N} + D_{F_N}I = 0$.

The operator D_F is symmetric on the sections with compact support in the interior of M. In order to define a self-adjoint extension of D_F we have to choose a suitable boundary condition. We will use a boundary condition of Atiyah-Patodi-Singer type that will depend on the choice of a Lagrangian subspace L in $V := ker(D_{F_N})$. Note that I acts on V. A subspace $L \subset V$ is a Lagrangian subspace if $L \oplus IL = V$ and the sum is orthogonal with respect to the scalar product induced by the L^2-metric. Let pr_L be the projection onto L and $E_{D_{F_N}}(.)$ be the spectral family of the unique self-adjoint extension of D_{F_N}. We define the initial domain of $D_{F,L}$ as

$$dom\ D_{F,L} = \{\psi \in C^\infty(M, \Lambda^{ev}T^*M \otimes F) \mid \left(E_{D_{F_N}}(-\infty, 0) + (1 - pr_L)\right)\psi_{|N} = 0\}.$$

Then $D_{F,L}$ is essentially self-adjoint and we denote the unique self-adjoint extension by the same symbol. We define the η-invariant of $D_{F,L}$ to be the real number

$$\eta(M, N, F, L) := \frac{1}{\pi}\int_0^\infty Tr D_{F,L} e^{-tD_{F,L}^2} t^{-1/2} dt.$$

The integral converges by the results of Douglas/Wojciechowski [4]. Equivalently, one could define the η-invariant as the value of the analytic continuation of the η-function

$$\eta(s) := Tr\frac{sign(D)}{|D|^s}, \quad Re(s) > n$$

at $s = 0$. We will also use the reduced η-invariant

$$\bar{\eta}(M, N, F, L) = \eta(M, N, F, L) - dim\, ker\, D_{F,L}.$$

The η-invariant is a global spectral invariant with a local variation formula, i.e. if δD_F is a local variation of the Dirac operator given by a variation of the flat bundle or the Riemannian metric supported in the interior of M the variation of the class $[\bar{\eta}(M, N, F, L)]$ modulo $2\mathbf{Z}$ is given by an integral

$$\delta([\bar{\eta}(M, N, F, L)]) = \int_M \Omega(D_F, \delta D_F),$$

where $\Omega(D_F, \delta D_F)$ is an n-form defined by a differential polynomial in the data defining $D_F, \delta D_f$.

The reduced η-invariant depends on the metric and the choice of the Lagrangian L. Thus, it is not canonically defined for a Riemannian manifold with boundary and a flat bundle. In the next sections we will alter the definition of the reduced η-invariant step by step making it more and more canonical and independent of the additional choices.

3 The canonical η-invariant of a manifold with boundary

In this section we explain how $\bar{\eta}(M, N, F, L)$ can be viewed as an element in the determinant line of the operator D_{F_N}. The author has learned this idea from X. Dai and was also inspired by the paper of Freed [5]. In fact, he independently found a similar construction using another complex line. After all, it turned out that this line, which is described in an appendix to the present section, is canonically isomorphic to the determinant line.

The finite-dimensional Hilbert space $V := ker\, D_{F_N}$ carries an action of the involution $-\imath I$ and splits into the corresponding ± 1 eigenspaces $V = V^+ \oplus V^-$. The determinant line of the operator D_{F_N} is defined as the one-dimensional complex Hilbert space

$$det(D_{F_N}) := det(V^+)^v \otimes det(V^-),$$

where $det(V^\pm)$ is the maximal alternating power of $V^\pm$. By convention, $det(\{0\})$ is taken to be $\mathbf{C}$.

A Lagrangian subspace $L \subset V$ defines an element $det(\sigma_L^+) \in det(D_{F_N})$ that we are going to describe now. Let $\sigma_L := pr_L - (1 - pr_L)$ be the reflection in

L. The involution σ_L anticommutes with I, i.e. $\sigma_L I + I\sigma_L = 0$, and splits into $\sigma_L^\pm : V^\pm \to V^\mp$. Then

$$det(-\imath\sigma_L^+) \in Hom(det(V^+), det(V^-)) = det(D_{F,N})$$

is the element induced by $-\imath\sigma_L^+$. In [3] we proved the following result (strengthening a previous result of Lesch/Wojciechowski [7]) about the dependence of the η-invariant on the choice of the Lagrangian subspace L.

Proposition 3.1 *Let $L_0, L_1 \subset V$ be Lagrangian subspaces. Then*

$$\bar\eta(M,N,F,L_0) - \bar\eta(M,N,F,L_1) = m(IL_1, L_0) - dim(IL_1 \cap L_0) \quad (mod\ 2\mathbf{Z}),$$

where the function m is defined on pairs of Lagrangian subspaces by

$$m(IL_1, L_0) = -\frac{1}{\pi} \sum_{e^{\imath\lambda} \in spec(\frac{\imath+I}{2\imath}\sigma_{L_1}\sigma_{L_0}), \lambda \in (-\pi,\pi)} \lambda.$$

The following proposition associates to the triple $(M,N,F) \in D$ equiped with Riemannian metrics an element $e(M,N,F) \in det(D_{F_N})$. We abbreviate $det(D_{F_N}) =: h(F,N)$.

Proposition 3.2 *The element $e(M,N,F) := e^{\pi\imath\bar\eta(M,N,F,L)} det(-\imath\sigma_L^+)$, defined for any Lagrangian subspace $L \subset V$, is independent of L.*

Proof: We will see that the L-dependence of $e^{\pi\imath\bar\eta(M,N,F,L)}$ cancels with that of $det(-\imath\sigma_L^+)$. In fact, using $(\sigma_L^+)^{-1} = \sigma_L^-$ we get

$$\begin{aligned}\frac{det(-\imath\sigma_{L_0}^+)}{det(-\imath\sigma_{L_1}^+)} &= det(\sigma_{L_1}^-) det(\sigma_{L_0}^+) \\ &= det(\sigma_{L_1}^- \sigma_{L_0}^+) \\ &= e^{-\imath\pi m(IL_1,L_0) + \imath\pi dim(IL_1 \cap L_0)}.\end{aligned}$$

□

Let $(f,\Phi) : (M_0,N_0,F_0) \to (M_1,N_1,F_1)$ be a morphism in the category D such that f is an isometry. Then $(\partial f, \partial\Phi) : (N_0,F_0) \to (N_1,F_1)$ induces an identification $\partial\Phi_* : V_0 \to V_1$ compatible with the involutions $-\imath I_j$, $j = 0,1$. Thus, we obtain an isometry

$$h(\partial\Phi, \partial f) := det((\partial\Phi_*^+)^{-1})^v \otimes det(\partial\Phi_*^-) : h(N_0,F_0) \to h(N_1,F_1)$$

satisfying

$$h(\partial\Phi,\partial f) e(M_0,N_0,F_0) = e(M_1,N_1,F_1).$$

Moreover, $Tr\, e(-M,-N,F) \otimes e(M,N,F) = (-1)^{dim(V)/2}$.

Appendix

We give another description of the determinant line $h(N, F)$. Consider the finite-dimensional Hilbert space $V := Ker(D_{F_N})$ together with the action of the complex structure I. Let $\Lambda(V, I)$ be the space (we need only the set without any manifold structure) of all Lagrangian subspaces of V. We consider $\Lambda(V, I)$ as the set of objects of a category Λ. In this category there is for any pair of objects L_0, L_1 exactly one morphism $L_0 \to L_1$. We define a functor $K : \Lambda \to W$. The functor K associates to every object L the complex line, i.e. $K(L) := \mathbf{C}$. For any morphism $L_0 \to L_1$ we let $K(L_0 \to L_1)$ be the multiplication with $e^{-\pi\imath(m(IL_1,L_0)-dim(IL_1\cap L_0))}$. Using a similar reasoning as in the proof of Proposition 3.2 one can check that this functor admits a one-dimensional space of sections $\Gamma(K)$. We claim $\Gamma(K) = h(N, F)$ in a canonical way. Let $L \subset V$ be a Lagrangian subspace. Then $det(\sigma_L^+) \in h(N, F)$ defines an identification $i_L : h(N, F) \to \mathbf{C}$ by $i_L(det(\sigma_L^+)) = 1$. The same Lagrangian subspace also defines an identification $j_L : \Gamma(K) \to \mathbf{C}$ by evaluating the section at L. For any two Lagrangian subspaces L_0, L_1 by the proof of 3.2 we have $i_{L_0} \circ j_{L_0}^{-1} = i_{L_1} \circ j_{L_1}^{-1}$. Thus, taking any Lagrangian $L \subset V$ we get a canonical isomorphism

$$j_L^{-1} \circ i_L : h(N, F) \to \Gamma(K).$$

The motivation for this construction is that $L \subset V \mapsto e^{\pi\imath\bar{\eta}(M,N,F,L)}$ can be viewed as a section of the functor K.

4 A difference construction

We are going to use a difference construction in order to avoid the metric dependence of $e(M, N, F)$. We have introduced the flat vector bundle F into the construction just to do this particular step. Let Det be the functor which associates to a closed oriented Riemannian manifold N of dimension $n-1$ equipped with a flat Hermitian vector bundle F the line

$$Det(N, F) := h(N, F) \otimes h(N)^{-dim(F)}.$$

If $(f, \Phi) : (N_0, F_0) \to (N_1, F_1)$ is a morphism in ∂D such that f is an isometry, then we have a natural map

$$Det(\Phi, f) := h(\Phi, f) \otimes (h(f)^{-1})^{\otimes -dim(F)} : Det(N_0, F_0) \to Det(N_1, F_1).$$

Let now $(M, N, F) \in D$ be equiped with a Riemannian metric g^N on N. We extend g^N to the interior of M and set

$$\epsilon(M, N, F)(g^N) := e(M, N, F) \otimes e(M, N)^{\otimes -dim(F)}.$$

Proposition 4.1 *The invariant $\epsilon(M,N,F)(g^N)$ is independent of the Riemannian metric in the interior of M extending the metric g^N on N.*

Proof: Note that the the space of all metrics on M extending a given metric g^N on the boundary N is linearly connected. Thus, it is enough to verify that $\epsilon(M,N,F)$ does not change under variations of the metric in the interior of M. Since F is flat, D_F is locally isomorphic to a direct sum of $dim(F)$ copies of D. By the local variation formula we obtain

$$\delta\epsilon(M,N,F)(g^N) = \imath\pi\epsilon(M,N,F)(g^N)\int_M (\Omega(D_F,\delta D_F) - dim(F)\Omega(D,\delta D)) = 0.$$

□

Let $(f,\Phi) : (M_0,N_0,F_0) \to (M_1,N_1,F_1)$ be a morphism in the category D such that $\partial f : N_0 \to N_1$ is an isometry. Then

$$Det(\Phi,f)\epsilon(M_0,N_0,F_0)(g^{N_0}) = \epsilon(M_1,N_1,F_1)(g^{N_1}). \tag{1}$$

Moreover,

$$Tr\,\epsilon(-M,-N,F)(g^N)\otimes\epsilon(M,N,F)(g^N) = (-1)^d, \tag{2}$$

where $2d = dim\,ker(D_{F_N}) + dim(F)dim\,ker(D_N)$.

5 The definition of H and E

In the last section we introduced a difference construction in order to make things independent of the Riemannian metric in the interior of (M,N). We will use a categorial construction in order to define an element of a line associated with the boundary, which is independent of the Riemannian metric of N.

Consider the category $Met(N,F)$. The objects of $Met(N,F)$ are the Riemannian metrics on N. For each pair of metrics g_0^N, g_1^N there is exactly one morphism $g_0^N \to g_1^N$ (a formal object). Recall, that W denotes the category of finite-dimensional complex Hilbert spaces. We define a functor $Z : Met(N,F) \to W$. Let $Z(g^N) := Det(N,F)(g^N)$. If $g_0^N \to g_1^N$ is a morphism, then we choose some path g_t^N of metrics connecting g_0^N and g_1^N. We require g_t^N to be constant near its ends. We set

$$\begin{aligned} Z(g_0^N \to g_1^N) &= \epsilon([0,1]\times N, \{0\}\times -N \cup \{1\}\times N, pr_N^*F)(g_0^N, g_1^N) \\ &\in Hom(Det(N,F)(g_0^N), Det(N,F)(g_1^N)). \end{aligned}$$

This definition is independent of the choice of the path by the result of the preceeding section. The only point to be verified is the composition rule.

Proposition 5.1 *Let g_i^N, $i = 0,1,2$ be Riemannian metrics on N. Then*

$$Z(g_0^N \to g_1^N \to g_2^N) = Z(g_1^N \to g_2^N)\circ Z(g_0^N \to g_1^N).$$

Proof: We employ the glueing formula for the η-invariant proved in [3]. Let (M_0, N_0, F_0) and (M_1, N_1, F_1) be oriented Riemannian manifolds with boundary with $N_0 = -N_1$ (isometrically) and $\Phi : F_{0|N_0} \to F_{1|N_1}$ be a given identification. We indicated only the boundary components, where the glueing takes place. There may be more boundary components, which are not affected by the glueing. We can glue at N obtaining a new manifold $M = M_0 \sharp_N M_1$, which may have boundaries together with a new flat bundle $F \to M$. Choose Lagrangian subspaces for every boundary component, in particular L_0 for N_0 and L_1 for N_1, in order to define the boundary conditions. Then we have proved in [3]

Theorem 5.2 (The glueing formula for the η-invariant)

$$\bar{\eta}(M, F) = \bar{\eta}(M_0, N_0, F_0, L_0) + \bar{\eta}(M_1, N_1, F_1, L_1) + m(L_0, L_1) + dim(L_0 \cap L_1) (mod 2\mathbf{Z}). \tag{3}$$

We apply this theorem in order to show

$$Tr\, e(M_0, N_0, F_0) \otimes e(M_1, N_1, F_1) = e(M, F) \tag{4}$$

(again we omit the other boundary components in the notation). We use the same Lagrangian $L_0, L_1 := \Phi(L_0)$ in order to define

$$e(M, N_0, F_0) = e^{\pi\imath\bar{\eta}(M_0,N_0,F_0,L_0)} det(-\imath\sigma^+_{L_0}(N_0))$$

$$e(M, N_1, F_1) = e^{\pi\imath\bar{\eta}(M_1,N_1,F_1,L_1)} det(-\imath\sigma^+_{L_1}(N_1)).$$

Under the identification $h(\Phi) : h(N_0, F_0) \to h(N_1, F_1)^v$ we have

$$Tr\, -\imath\sigma^+_{L_0}(N_0) \otimes -\imath\sigma^+_{L_1}(N_1) = (-1)^{dim(V)/2}.$$

This sign cancels with the $e^{\pi\imath dim(L_0\cap\Phi^{-1}(L_1))}$ of (3) and $m(L_0, \Phi^{-1}(L_1)) = 0$. Thus, we obtain (4). Applying the same argument to the trivial bundle we obtain

Corollary 5.3

$$Tr\, \epsilon(M_0, N_0, F_0)(g^N) \otimes \epsilon(M_1, N_1, F_1)(g^N) = \epsilon(M, F).$$

Proposition 5.1 immediately follows from Corollary 5.3. □

The category $Met(N, F)$ is connected and the functor Z has trivial holonomy. Consider the cylinder

$$Cyl := ([0, 1] \times N, \{0\} \times -N \cup \{1\} \times N, pr^*_N F, (dr)^2 \oplus g^N),$$

where we use the constant path of metrics given by g^N. We have to show

$$\epsilon(Cyl)(g^N, g^N) = id \in Hom(Det(N, F), Det(N, F)).$$

This can be seen by glueing the boundary components using 5.3 and

$$\epsilon(S^1 \times N, pr_N^* F) = 1.$$

Now we define the functor $H : \partial D \to W$ by $H(N, F) := \Gamma(Z)$. Consider a morphism $(\Phi, f) : (N_0, F_0) \to (N_1, F_1)$ in the category ∂D. We have to define $H(\Phi, f) : H(N_0, F_0) \to H(N_1, F_1)$. Choose a Riemannian metric g^{N_1} on N_1 and take $g^{N_0} := f^* g^{N_1}$. Then f becomes an isometry. The choice of the Riemannian metrics fixes isomorphisms $i_k : H(N_k, F_k) \to Det(N_k, F_k)$, $k = 0, 1$. Moreover there is a natural isomorphism $Det(\Phi, f) : Det(N_0, F_0) \to Det(N_1, F_1)$. Letting $H(\Phi, f) := i_1^{-1} \circ Det(\Phi, f) \circ i_0$ it is easy to see that this definition does not depend on the choice of the Riemannian metric g^{N_1}.

For any object (M, N, F) of D we will now define the vector $E(M, N, F) \in H(N, F)$. Here we use the following fact.

Proposition 5.4 *Let $(M, N, F) \in D$. The correspondence*

$$Met(N, F) \ni g^N \mapsto \epsilon(M, N, F)(g^N) \in Z(g^N)$$

defines a section $E(M, N, F) \in H(N, F)$.

Proof: Let $g_0^N \to g_1^N$ be a morphism of $Met(N, F)$. Let g_t^N be a path between g_0 and g_1 being constant near its ends. Then we must verify that

$$Tr\epsilon(M, N, F)(g_0^N) \otimes \epsilon([0, 1] \times N, \{0\} \times -N \cup \{1\} \times N, pr_N^* F)(g_0^N, g_1^N) = \epsilon(M, N, F)(g_1^N).$$

But this follows from Corollary 5.3. □

It is easy to check that $(M, N, F) \mapsto E(M, N, F)$ is a section of the functor $H \circ \partial : D \to W$.

Theorem 5.5 *The functor $H : \partial D \to W$ and the section $E \in \Gamma(H \circ \partial)$ define a local Lagrangian of a TQFT on the category D.*

Proof: The orientation axiom (up to a sign) follows from (2). The additivity axiom is obvious and locality follows once more from Corollary 5.3. □

What we have constructed, is a classical TQFT. The next step would be to "take sums" over the set of flat bundles. This is the so-called second quantization (see Freed/Quinn [6]). This step is much more complicated. Our theory is very similar to the TQFT's, which have already been constructed using the Chern-Simons action. Both theories have similar infinitesimal variation formulas. While for the Chern-Simons gauge theory the values on the closed manifolds are the values of the Chern-Simons functional of the flat connections, in our case the values are the ξ-invariants of Atiyah-Patodi-Singer [2], which are, in fact, also a sort of Chern-Simons invariants. The difference of the theory defined with the η-invariant and the Chern-Simons theory is potentially located in its contents for manifolds with non-empty boundaries. For the Chern-Simons action the second quantization has been carried out by Freed/Quinn [6] for a finite gauge group. æ

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